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Analysis of the Order Parameter for Uniform Nearest Particle System

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The uniform nearest particle system (UNPS) is studied, which is a continuous-time Markov process with state space $\{0, 1\}^{Z^1}$. The rigorous upper bound $\rho_\lambda^{(mf)} = (\lambda - 1)/\lambda$ for the order parameter ρ_λ is given by the correlation identity and the FKG inequality. Then an improvement of this bound $\rho_\lambda^{(mf)}$ is shown in a similar fashion; $\rho_\lambda \leqslant C(\lambda - 1)/|\log(\lambda - 1)|$ for $\lambda > 1$. Recently, Mountford proved that the critical value $\lambda_c = 1$. Combining his result and our improved bound implies that if the critical exponent β exists, it is strictly greater than the mean-field value 1 in the weak sense.

KEY WORDS: Uniform nearest particle system; order parameter; critical value; critical exponent; correlation identities; the FKG inequality.

1. INTRODUCTION

The uniform nearest particle system (UNPS) is a continuous time Markov process with state space $X \equiv \{0, 1\}^{Z^1}$ (see Chapter VII of ref. 1). It may be thought that, for $\eta \in X$, x is vacant if $\eta(x) = 0$, and x is occupied by a particle if $\eta(x) = 1$ at each site x. The formal generator Ω is

$$\Omega f(\eta) = \sum_{x \in Z^1} \{\eta(x) + \beta_\lambda(l_x(\eta), r_x(\eta))[1 - \eta(x)]\}\{f(\eta_x) - f(\eta)\}$$

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where λ is a nonnegative parameter,

$$\beta_\lambda(l, r) \equiv \frac{\lambda}{l+r-1} \qquad (l, r \geqslant 1)$$

$$l_x(\eta) \equiv x - \max\{y < x \mid \eta(y) = 1\}$$

$$r_x(\eta) \equiv \min\{y > x \mid \eta(y) = 1\} - x$$

$\eta \in X$, and $\eta_x \in X$ is defined by $\eta_x(y) = \eta(y)$ if $y \neq x$ and $\eta_x(x) = 1 - \eta(x)$.

In this system, the addition of an extra occupied site to the state of the process does not decrease the rate at which any of the vacant sites become occupied, and the occupied sites become vacated with a constant rate 1. In this sense, the UNPS is an attractive spin system. However it is not a reversible system (see Chapter VII, Section 4 of ref. 1 for the conditions of the reversible nearest particle systems).

Let δ_i denote the point mass on $\eta \equiv i$ $(i = 0, 1)$. The fact that $\delta_0 = \delta_0 S(t)$ $(t \geqslant 0)$ implies obviously that the lower invariant measure $\lim_{t \to \infty} \delta_0 S(t)$ equals δ_0, where $S(t)$ is the semigroup corresponding to Ω.

On the other hand, the upper invariant measure $\nu_\lambda \equiv \lim_{t \to \infty} \delta_1 S(t)$ is dependent on the value λ $(\geqslant 0)$. It is known that $\nu_\lambda = \delta_0$ for $\lambda \leqslant 1$ (see Theorem 5.5, Chapter VII of ref. 1.) Recently, Mountford[2] obtained that $\nu_\lambda \neq \delta_0$ for $\lambda > 1$ in the following more general class of the birth rates $\beta_{\lambda, f}(l, r)$ for $\lambda \geqslant 0$ and $f \in \Phi$, where

$$\Phi \equiv \Bigg\{ f\colon [0, 1] \to [0, \infty) \mid f(x) = f(1-x) \text{ for any } x \in [0, 1],$$

$$\int_0^1 f(x)\, dx = 1,\ xf(x) \text{ is a nondecreasing function on } [0, 1],$$

$$\text{there is a strictly positive } \alpha \text{ such that } \int_{0+}^1 x^{-\alpha} f(x)\, dx < \infty \Bigg\}$$

and

$$\beta_{\lambda, f}(l, r) \equiv \frac{\lambda}{l+r-1} \int_{l-1}^{l} f\left(\frac{x}{l+r-1}\right) dx$$

for any $\lambda \geqslant 0$ and $f \in \Phi$. In particular, $f \equiv 1$ $(\in \Phi)$ is in the case of the UNPS.

We define ρ_λ as the expectation with respect to ν_λ of $\eta(x)$ for a site $x \in Z^1$,

$$\rho_\lambda \equiv E_{\nu_\lambda}[\eta(x)]$$

Note that ρ_λ is independent of $x \in Z^1$. By the attractiveness of the UNPS, ρ_λ is a $[0, 1]$-valued nondecreasing function of λ ($\geqslant 0$). We take ρ_λ as an *order parameter* of the UNPS. The *critical value* $\lambda_c = 1$ is characterized by

$$\lambda_c \equiv \inf\{\lambda \geqslant 0 \mid \rho_\lambda > 0\}$$

In this paper, we define a *critical exponent* β (>0) of ρ_λ by

$$\rho_\lambda \sim (\lambda - \lambda_c)^\beta \qquad \text{for} \quad \lambda \gtrsim \lambda_c$$

where $f(\lambda) \sim g(\lambda)$ for $\lambda \gtrsim \lambda_c$ means $\log f(\lambda)/\log g(\lambda) \to 1$ as $\lambda \downarrow \lambda_c$. Furthermore, the critical exponent β of ρ_λ is said to be strictly greater (resp. less) than $\tilde{\beta}$ (>0) *in a weak sense* if

$$\lim_{\lambda \downarrow \lambda_c} \frac{\rho_\lambda}{(\lambda - \lambda_c)^{\tilde{\beta}}} = 0 \qquad (\text{resp. } \infty)$$

On the other hand, the critical exponent β of ρ_λ is said to be strictly greater (resp. less) than $\tilde{\beta}$ (>0) *in a strong sense* if

$$\lim_{\lambda \downarrow \lambda_c} \frac{\log \rho_\lambda}{\log(\lambda - \lambda_c)^{\tilde{\beta}}} > 1 \qquad (\text{resp. } < 1)$$

The objective of the present paper is to give the rigorous upper bound for ρ_λ and information on the critical exponent of ρ_λ as λ approaches λ_c. The analysis of the critical exponent has played an important role in mathematical physics.

In Section 2, correlation identities of the UNPS are given. In Section 3, the mean-field upper bound $\rho_\lambda^{(mf)} = (\lambda - 1)/\lambda$ for ρ_λ is obtained by using the correlation identity and the FKG inequality. Then $\rho_\lambda^{(mf)}$ gives the critical exponent $\beta^{(mf)} = 1$ of the mean-field type. Combining this bound and Mountford's result $\lambda_c = 1$, it is concluded that the second-order phase transition appears in the UNPS. Furthermore, a similar procedure yields that $\rho_\lambda \leqslant C(\lambda - 1)/|\log(\lambda - 1)|$ for $\lambda > 1$, in Section 4. That is to say, if the critical exponent β of ρ_λ exists, then it is strictly greater than the mean-field critical exponent $\beta^{(mf)} = 1$ in the weak sense. Section 5 is devoted to summary and discussions.

2. CORRELATION IDENTITIES

We introduce the correlation identities in the UNPS which will be used in the next two sections. Let $K(n_1, n_2, \cdots, n_N)$ be the probability of having 1's at 0, n_1+1, $n_1+n_2+2, \cdots, n_1+\cdots+n_N+N$ and 0's at all other sites in $[0, n_1+\cdots+n_N+N]$,

$$K(n_1, n_2, \cdots, n_N) = E_{\nu_\lambda}\left[\eta(0) \prod_{i=1}^{N} \left\{\prod_{k_i=1}^{n_i} [1-\eta(n_1+\cdots+n_{i-1}+k_i+i-1)] \times \eta(n_1+\cdots+n_i+i)\right\}\right]$$

for any $N=1, 2, \cdots$ and $n_1, n_2, \cdots, n_N \in \{0, 1, \cdots\}$.

It is easy to obtain the following correlation identities in the UNPS.[3]

Lemma 2.1. In the UNPS, for $\lambda > \lambda_c$,

(i) $-\rho_\lambda + \lambda \sum_{k=1}^{\infty} K(k) = 0$

(ii) $-K(0) + \lambda \sum_{k=1}^{\infty} \frac{K(k)}{k} = 0$

(iii) $-(\lambda+2)\, K(n) + \sum_{k=1}^{n} K(k-1, n-k) + 2\lambda \sum_{k=1}^{n} \frac{K(k+n)}{k+n} = 0 \qquad (n=1, 2, \cdots)$

3. MEAN-FIELD BOUND

In this section, we give a mean-field bound $\rho_\lambda^{(mf)}$ for ρ_λ by the correlation identity and the FKG inequality. Let $Y \equiv \{A \subset \mathbf{Z}^1 \mid \#(A) < \infty\}$, where $\#(A)$ is the cardinality of A. Then define, for any $A \in Y$,

$$\rho(A) \equiv E_{\nu_\lambda}\left[\prod_{j \in A} \eta(j)\right]$$

and

$$\bar{\sigma}(A) \equiv E_{\nu_\lambda}\left\{\prod_{j \in A} [1-\eta(j)]\right\}$$

In particular, $\rho(\{0\}) = \rho_\lambda$ and $\bar{\sigma}(\{0\}) = 1 - \rho_\lambda$.

For $\lambda > \lambda_c$, a simple calculation yields

$$\sum_{k=1}^{\infty} K(k) = \rho_\lambda - \rho(\{0, 1\}) \tag{3.1}$$

Therefore, by Lemma 2.1(i) and (3.1), we have

$$(\lambda-1)\,\rho_\lambda-\lambda\rho(\{0,1\})=0 \tag{3.2}$$

On the other hand, the FKG inequality gives the following lemma (for example, see Theorem 2.13, Chapter III of ref. 1. An application of the FKG inequality to the contact processes was shown in ref. 4):

Lemma 3.1. For any $A, B \in Y$,

(i) $\rho(A \cup B) \geqslant \rho(A)\,\rho(B)$

(ii) $\bar{\sigma}(A \cup B) \geqslant \bar{\sigma}(A)\,\bar{\sigma}(B)$

Applying Lemma 3.1(i) to (3.2) yields

$$\rho_\lambda\left(\rho_\lambda-\frac{\lambda-1}{\lambda}\right)\leqslant 0 \tag{3.3}$$

Hence, we give the following result.

Theorem 3.2. For $\lambda \geqslant 1$,

$$\rho_\lambda \leqslant \frac{\lambda-1}{\lambda} \tag{3.4}$$

In this way, the rigorous upper bound $\rho_\lambda^{(mf)} \equiv (\lambda-1)/\lambda$ $(\lambda \geqslant 1)$ can be easily obtained. Fortunately, Mountford showed $\lambda_c = 1$, so the mean-field bound $\rho_\lambda^{(mf)}$ implies that the second-order phase transition appears in the UNPS. In addition, it has mean-field critical exponent $\beta^{(mf)} = 1$. Furthermore, by using (3.4), we have immediately the next corollary.

Corollary 3.3.

$$\lim_{\lambda \downarrow 1} \frac{\log \rho_\lambda}{\log(\lambda-1)} \geqslant 1 \tag{3.5}$$

This result means that if $\rho_\lambda \sim (\lambda-1)^\beta$ for $\lambda \gtrsim 1$, then $\beta \geqslant 1$. In the next section, more detailed information on β will be given.

Remark 3.4. It is easily obtained that (3.4) is satisfied in the more general setting of the birth rates $\beta_{\lambda,f}(l,r)$ (for any $f \in \Phi$) defined in Section 1.

4. IMPROVED BOUND FOR CRITICAL EXPONENT

To begin, Lemma 2.1(ii) is rewritten as

$$K(0)=\lambda \sum_{k=1}^{\infty} \frac{K(k)}{k} \tag{4.1}$$

The left side of (4.1) can be written as

$$1-2\bar{\sigma}(\{0\})+\bar{\sigma}(\{0,1\}) \tag{4.2}$$

In a similar fashion, the right side of (4.1) can be written as

$$\begin{aligned}&\lambda \sum_{k=1}^{\infty} \frac{1}{k}\{\bar{\sigma}(\{0,1,\cdots,k-1\})-2\bar{\sigma}(\{0,1,\cdots,k\})+\bar{\sigma}(\{0,1,\cdots,k+1\})\}\\ &\quad=\lambda\bar{\sigma}(\{0\})-\frac{3}{2}\lambda\bar{\sigma}(\{0,1\})+2\lambda\sum_{k=1}^{\infty}\frac{\bar{\sigma}(\{0,1,\cdots,k+1\})}{k(k+1)(k+2)}\end{aligned} \tag{4.3}$$

Combining (4.2) and (4.3) yields

$$-1+(\lambda+2)\,\bar{\sigma}(\{0\})-\left(\frac{3}{2}\lambda+1\right)\bar{\sigma}(\{0,1\})+2\lambda\sum_{k=1}^{\infty}\frac{\bar{\sigma}(\{0,1,\cdots,k+1\})}{k(k+1)(k+2)}=0 \tag{4.4}$$

By Lemma 2.1(i), we have

$$\bar{\sigma}(\{0,1\})=\frac{(\lambda+1)\,\bar{\sigma}(\{0\})-1}{\lambda} \tag{4.5}$$

Applying (4.5) and $\bar{\sigma}(\{0\})=1-\rho_\lambda$ to (4.4) gives the following lemma.

Lemma 4.1. For $\lambda>1$,

$$\rho_\lambda=\frac{\lambda^2}{\lambda^2+\lambda+2}\left\{1-4\sum_{k=1}^{\infty}\frac{\bar{\sigma}(\{0,1,\cdots,k+1\})}{k(k+1)(k+2)}\right\} \tag{4.6}$$

It is remarked that Lemma 4.1 can be obtained by Lemma 2.1(iii) instead of Lemma 2.1(ii). Using Lemma 3.1(ii) repeatedly, we obtain

$$\bar{\sigma}(\{0,1,\cdots,k+1\})\geqslant\bar{\sigma}(\{0\})^{k+2} \qquad (k=1,2,\cdots) \tag{4.7}$$

so that

$$\rho_\lambda\leqslant\frac{\lambda^2}{\lambda^2+\lambda+2}\left\{1-4\sum_{k=1}^{\infty}\frac{\bar{\sigma}(\{0\})^{k+2}}{k(k+1)(k+2)}\right\} \tag{4.8}$$

On the other hand, it is easily obtained that for $x \in [0, 1]$

$$\sum_{k=1}^{\infty} \frac{x^{k+2}}{k(k+1)(k+2)} = -\frac{1}{2}(1-x)^2 \log(1-x) + \frac{3}{4}x^2 - \frac{x}{2} \qquad (4.9)$$

By (4.8), (4.9), and $\bar{\sigma}(\{0\}) = 1 - \rho_\lambda$,

$$\rho_\lambda \leqslant \frac{\lambda^2}{\lambda^2 + \lambda + 2}\{2\rho_\lambda^2 \log \rho_\lambda + 4\rho_\lambda - 3\rho_\lambda^2\}$$

Therefore, we have the next main result.

Theorem 4.2. For $\lambda > 1$,

$$\frac{\rho_\lambda}{\lambda - 1} \leqslant \frac{3\lambda + 2}{\lambda^2(3 - 2\log \rho_\lambda)} \qquad (4.10)$$

By the above theorem, we have more detailed information on the critical behavior of ρ_λ.

Corollary 4.3.

$$\lim_{\lambda \downarrow 1} \rho_\lambda \frac{|\log(\lambda - 1)|}{\lambda - 1} \leqslant \frac{5}{2} \qquad (4.11)$$

This result implies that there exists a constant $C > 0$ such that for $\lambda > 1$,

$$\rho_\lambda \leqslant C \frac{\lambda - 1}{|\log(\lambda - 1)|} \qquad (4.12)$$

From (4.12), it follows that if the critical exponent β exists, then it is strictly greater than the mean-field critical exponent $\beta^{(mf)} = 1$ in the weak sense.

Remark 4.4. Let us define

$$\tilde{\Phi} \equiv \left\{ f \in \Phi \,\middle|\, \Delta F\left(\frac{1}{n+1}\right) \geqslant \frac{1}{n(n+1)(n+2)} \quad \text{for} \quad n = 1, 2, \cdots \right\}$$

where, for any $f \in \Phi$ and $n = 1, 2, \cdots$,

$$F\left(\frac{1}{n}\right) \equiv \int_0^{1/n} f(x)\, dx$$

and

$$\Delta F\left(\frac{1}{n+1}\right) \equiv \frac{1}{2}\left\{F\left(\frac{1}{n}\right) - 2F\left(\frac{1}{n+1}\right) + F\left(\frac{1}{n+2}\right)\right\}$$

It is easily checked that (4.10) is satisfied in the nearest particle system with the birth rates $\beta_{\lambda,f}(l, r)$ for any $f \in \tilde{\Phi}$. In the case of the UNPS, i.e., $f \equiv 1$, then

$$\Delta F\left(\frac{1}{n+1}\right) = \frac{1}{n(n+1)(n+2)}$$

so $f \equiv 1 \in \tilde{\Phi}$.

5. SUMMARY AND DISCUSSION

The order parameter ρ_λ in the UNPS has been studied in the present paper from the mathematical point of view. By the correlation identity and the FKG inequality, we obtained the mean-field upper bound $\rho_\lambda^{(mf)} = (\lambda - 1)/\lambda$ for ρ_λ. (See Theorem 3.2.) $\rho_\lambda^{(mf)}$ has the critical exponent $\beta^{(mf)} = 1$ of the mean-field type. Mountford showed $\lambda_c = 1$ recently. Therefore, if $\rho_\lambda \sim (\lambda - \lambda_c)^\beta$ for $\lambda \gtrsim \lambda_c$, then $\beta \geqslant 1$.

A similar procedure gave further information on the critical behavior of ρ_λ. Corollary 4.3 implies that if the critical exponent β of ρ_λ exists, then it is strictly greater than the mean-field critical exponent $\beta^{(mf)} = 1$ in the weak sense. This result contrasts with the ordinary percolation model, for which Kesten and Zang[5] proved that the critical exponent for percolation probability in two dimensions is strictly less than one in the strong sense.

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REFERENCES

1. T. M. Liggett, *Interacting Particle Systems* (Springer-Verlag, New York, 1985).
2. T. S. Mountford, The critical value for the uniform nearest particle process, Preprint (1990).
3. N. Konno and M. Katori, Correlation identities for nearest-particle systems and their applications to one-dimensional contact process, *Mod. Phys. Lett. B* **5**:151–159 (1991).
4. N. Konno and M. Katori, Applications of the Harris-FKG inequality to upper bounds for order parameters in the contact processes, *J. Phys. Soc. Jpn.* **60**:430–434 (1991).
5. H. Kesten and Y. Zang, Strict inequalities for some critical exponents in two-dimensional percolation, *J. Stat. Phys.* **46**:1031–1055 (1987).